Qualitative Properties of Scalar-Tensor Theories of Gravity

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Abstract

The qualitative properties of spatially homogeneous stiff perfect fluid and minimally coupled massless scalar field models within general relativity are discussed. Consequently, by exploiting the formal equivalence under conformal transformations and field redefinitions of certain classes of theories of gravity, the asymptotic properties of spatially homogeneous models in a class of scalar-tensor theories of gravity that includes the Brans-Dicke theory can be determined. For example, exact solutions are presented, which are analogues of the general relativistic Jacobs stiff perfect fluid solutions and vacuum plane wave solutions, which act as past and future attractors in the class of spatially homogeneous models in Brans-Dicke theory.

1 Introduction

Scalar-tensor theories of gravitation, in which gravity is mediated by a long-range scalar field in addition to the usual tensor fields present in Einstein's theory, are the most natural alternatives to general relativity (GR). Scalar-tensor theories of gravity were originally motivated by the desire to incorporate a varying Newtonian gravitational constant, G, into GR, where a varying G was itself postulated for a variety of observational and theoretical reasons (cf. Barrow, 1996). Indeed, the simplest Brans-Dicke theory of gravity (BDT; Brans and Dicke, 1961), in which a scalar field, ϕ, acts as the source for the gravitational coupling with $G \sim \phi^{-1}$, was essentially motivated by apparent discrepancies between observations and the weak-field predictions of GR. More general scalar-tensor theories with a non-constant BD parameter, $\omega(\phi)$, and a non-zero self-interaction scalar potential, $V(\phi)$, have been formulated, and the solar system and astrophysical constraints on these theories, and particularly on BDT, have been widely studied (Will, 1993; see also Barrow and Parsons, 1997). Observational limits on the present value of ω_0 need not constrain the value of ω at early times in

more general scalar-tensor theories (than BDT). Hence, more recently there has been greater focus on the early Universe predictions of scalar-tensor theories of gravity, with particular emphasis on cosmological models in which the scalar field acts as a source for inflation (La and Steinhardt, 1989; Steinhardt and Accetta, 1990).

There are many exact cosmological solutions known in BDT. The earliest flat isotropic and homogeneous Friedmann-Robertson-Walker (FRW) exact BDT solutions presented were the vacuum solutions of O'Hanlon and Tupper (1972) and the special class of power-law perfect fluid solutions of Nariai (1968) with $p = (\gamma - 1)\rho$, where γ is a constant. The general solutions can be found for all γ; exact zero-curvature solutions were given by Gurevich et al. (1973) and the curved FRW models were presented by Barrow (1993) (these solutions are surveyed in Holden and Wands, 1998).

A phase-space analysis of the class of FRW models was performed by Kolitch and Eardley (1993) and was improved upon by Holden and Wands (1998) who presented all FRW models in a single phase plane (including those at "infinite" values via compactification). It was found that typically at early times ($t \rightarrow 0$) the BDT solutions are approximated by vacuum solutions (i.e., the O'Hanlon-Tupper FRW vacuum solutions) and at late times ($t \rightarrow \infty$) by matter-dominated solutions, in which the matter is dominated by the BD scalar field (e.g., the power-law Nariai solutions). Particular attention was focussed on whether inflation occurs and whether models have an initial singularity.

A variety of exact spatially homogeneous but anisotropic BDT solutions have also been found (Brans and Dicke, 1961; Nariai, 1972; Belinskii and Khalatnikov, 1973; Ruban, 1977; Lorenz-Petzold, 1984 and citations therein, Mimosa and Wands, 1995b). Various partial results concerning the asymptotic behaviour of Bianchi models in BDT have also been discussed. For example, Chauvet and Cervantes-Cota (1995) studied the possible isotropization of special classes of Bianchi models and Guzman (1997) presented a proof of the cosmic-no-hair theorem for ever-expanding spatially homogeneous BDT models with matter and a positive constant vacuum energy-density. However, there is no comprehensive and definitive discussion of the qualitative properties of anisotropic models in BDT.

Exact perfect fluid solutions in scalar-tensor theories of gravity with a non-constant BD parameter $\omega(\phi)$ have been obtained by various authors; the isotropic FRW vacuum and radiation solutions of Barrow (1993), which utilized the techniques of Lorenz-Petzold (1984), were generalized in the zero-curvature case to the more general perfect fluid case with a linear barotropic equation of state (satisfying $0 \leq \gamma \leq 4/3$ but including the important case of dust $\gamma = 0$) by Barrow and Mimosa (1994) and to stiff fluids in addition to vacuum and radiation for curved models by Mimosa and Wands (1995a). A variety of inflationary and non-inflationary solutions were obtained. This work was extended in a systematic study of the qualitative analysis of curved FRW models with a specific form for $\omega(\phi)$ by Barrow and Parsons (1997); in particular, the question of whether a given scalar-tensor

theory solution can approach GR in the weak-field limit at late times was addressed. This work was further generalized by Mimosa and Wands (1995b) to various special anisotropic Bianchi models for both BDT and scalar-tensor theories with a particular form for $\omega(\phi)$; both exact solutions were obtained and the asymptotic limits of the solutions, including their possible isotropization, were studied. The qualitative properties of both isotropic and special anisotropic scalar-tensor theory models was also studied by Serna and Alimi (1996). Isotropization and inflation in anisotropic scalar-tensor theories was discussed earlier by Pimentel and Stein-Schabes (1989). In summary, there exist a multitude of partial results on the possible qualitative behaviour of cosmological models in scalar-tensor theories, where the details of their asymptotic properties depend on the particular functional form of $\omega(\phi)$ assumed [for more references see Wands and Mimosa (1995b) and other papers cited above].

Scalar-tensor theories with a "free" scalar field are perhaps not well motivated since, often, quantum corrections produce interactions resulting in a non-trivial potential $V(\phi)$. More general scalar-tensor theories including a non-zero scalar potential, and in particular their qualitative properties, have also been studied (see Billyard et al, 1998, and references therein).

Scalar-tensor theory gravity is currently of great interest particularly since such theories occur as the low-energy limit in supergravity theories from string theory (Green et al., 1988) and other higher-dimensional gravity theories (Applequist et al., 1987). Indeed, superstring theory is currently the favoured candidate for a unified theory of the fundamental interactions that include gravity, and as such ought to describe the evolution of the very early Universe. In fact BDT, which is the simplest scalar-tensor theory, originated from taking seriously the scalar field arising in Kaluza-Klein compactification of the fifth dimension. Superstring theory leads to a variety of new cosmological possibilities including the so-called 'pre-big-bang' scenario (Veneziano, 1991; Gasparini and Veneziano, 1993), and cosmology is the ideal setting in which to study possible stringy effects.

Lacking a full non-perturbative formulation which allows a description of the early Universe close to the Planck time, it is necessary to study classical cosmology prior to the GUT epoch by utilizing the low-energy effective action induced by string theory. To lowest order in the inverse string tension the tree-level effective action in four-dimensions for the massless fields includes the non-minimally coupled graviton, the scalar dilaton and an antisymmetric rank-two tensor, hence generalizing GR (which is presumably a valid description at late, post-GUT, epochs) by including other massless fields. Additional fields, depending on the particular superstring model, are negligible in this low-energy limit and can be assumed to be frozen, and hence the massless bosonic sector of (heterotic) string theory reduces generically to a four-dimensional scalar-tensor theory of gravity. As a result, BDT includes the dilaton-graviton sector of the string effective action as a special case ($\omega = -1$) (Green et al., 1987). [1]

[1] Although this result is strictly only true in the absence of coupling to other matter fields, it remains

A variety of exact string-dilaton cosmological solutions have been found. These include spatially homogeneous models (both Bianchi and Kantowski-Sachs models and their isotropic specializations) and more recently inhomogeneous models (see Barrow and Kunze, 1998, and Lidsey, 1998, and references within). In addition, Clancy et al. (1998) have begun an investigation of the qualitative properties of a class of anisotropic Bianchi models within the context of four-dimensional low-energy effective bosonic string theory. Applications to string theory of techniques developed to study scalar-tensor gravity have been discussed in the isotropic case by Copeland et al. (1994) and in the anisotropic case by Mimosa and Wands (1995b).

As noted above these results are only partial results obtained by treating various special cases. In this paper we shall extend this work and present results on the general asymptotic properties of spatially homogeneous cosmological models in BDT (and in more general scalar-tensor theories of gravity). To our knowledge the only previous generic results in BDT are the investigation of the asymptotic character of solutions close to the cosmological singularity by Belinskii and Khalatnikov (1973), the study of mixmaster behaviour by Carretero-Gonzales et al. (1994) and the cosmic-no-hair theorem results of Guzman (1997).

In the next section we establish the formal equivalence between stiff perfect fluid models in GR and cosmological models in a class of scalar-tensor theories of gravity (including BDT) under conformal transformations and field redefinitions. In section 3 we then discuss the known asymptotic properties of stiff perfect fluid models in GR; summarizing these results:

1. For all models (Bianchi models of classes A and B), a subset of the Jacobs Disc, which consists of exact self-similar *Jacobs* stiff fluid solutions (corresponding to singular points of the governing system of autonomous ordinary differential equations), is the *past* attractor.
2. As regards *future* evolution, all stiff models behave like *vacuum* models with the following exceptions:
   (i) Bianchi I models, all of which are exact Jacobs solutions.
   (ii) Bianchi II models, which are future asymptotic to another subset of the Jacobs Disc.

   For Bianchi models of types VI_0 and VII_0 the future asymptote is a flat Kasner model, as in the case of vacuum models. The Bianchi VIII models do not have a self-similar future asymptote; these ever-expanding stiff models are the only models for which this is the case.

valid at least for the massless fields appearing in the low-energy effective string action.

In section 3 we also discuss stiff perfect fluid models in GR with an additional non-interacting perfect fluid or a cosmological constant. Massless scalar field models in GR are subsequently discussed. In section 4 we discuss the qualitative properties of spatially homogeneous models in a class of scalar-tensor theories of gravity. This is done by exploiting the formal equivalence of these theories with GR and utilizing the results of section 3. We shall concentrate on BDT. In particular, we shall present some exact BDT solutions, including analogues of the general relativistic Jacobs stiff perfect fluid solutions and vacuum solutions (and especially a Bianchi type VII_h plane wave solution) alluded to above, which act as past and future attractors in the class of spatially homogeneous BDT models. The asymptotic properties of models in the class of scalar-tensor theories of gravity under consideration can then be easily determined. The qualitative properties of more general scalar-tensor theories, including those with a non-zero scalar potential, can be studied in a similar way (cf. Billyard et al., 1998).

2 Analysis

A class of scalar-tensor theories, formally equivalent under appropriate conformal transformations and field redefinitions, are given by the action (in the *Jordan* frame) (cf. Mimosa and Wands, 1995b)

$$\overline{S} = \int \sqrt{-\overline{g}} \left[\phi \overline{R} - \frac{\omega(\phi)}{\phi} \overline{g}^{ab} \phi_{,a} \phi_{,b} + 2\overline{L}_m \right] d^4x \tag{2.1}$$

where L_m is the Lagrangian for the matter fields, which we shall assume corresponds to a comoving (i.e., the velocity of matter, u^a, is parallel to the unit normal to the spatial hypersurface) perfect fluid with energy density $\overline{\rho}$ and pressure $\overline{p}$. The main purpose of this paper is to study the asymptotic properties of spatially homogeneous models in this class of theories. In particular, we are interested in the BDT case in which $\omega(\phi) = \omega_0$, where ω_0 is a constant.

Under the conformal transformation and field redefinition

$$g_{ab} = \phi \overline{g_{ab}} \quad (dt = \pm\sqrt{\phi} d\overline{t}) \tag{2.2}$$

$$\frac{d\varphi}{d\phi} = \frac{\pm\sqrt{\omega(\phi) + 3/2}}{\phi} \tag{2.3}$$

the action becomes (in the *Einstein* frame)

$$S = \int \sqrt{-g}[R - g^{ab}\varphi_{,a}\varphi_{,a} + 2L_m] d^4x, \tag{2.4}$$

where

$$L_m = \frac{\overline{L}_m}{\phi^2}. \tag{2.5}$$

The action S is equivalent to the action for GR minimally coupled to a massless scalar field φ and matter (L_m). We shall attempt to exploit this equivalence to study the asymptotic properties of the scalar-tensor theories of gravity with action (2.1). In the spatially homogeneous case under consideration $\phi = \phi(t)$, and hence under the transformation (2.2) the Bianchi type of the underlying model is invariant. Also, in all applications here the transformation (2.2) is non-singular and so the asymptotic behaviour of the scalar-tensor theories (2.1) can be determined directly from the corresponding behaviour of the GR models (cf. Billyard et al., 1998)

In the scalar-tensor theory (2.1), the energy-momentum of the matter fields is separately conserved. In the Einstein frame this is no longer the case (although the overall energy-momentum of the combined scalar field and matter field is, of course, conserved). Indeed (Mimosa and Wands, 1995b),

$$\nabla^a T_{ab} = -\frac{1}{2}\frac{\phi_{,b}}{\phi} T^a_a. \tag{2.6}$$

Defining

$$\begin{aligned} \rho &= \frac{\overline{\rho}}{\phi^2}, \\ p &= \frac{\overline{p}}{\phi^2}, \end{aligned} \tag{2.7}$$

then in the spatially homogeneous case we obtain the conservation equation

$$\dot{\rho} + 3(\rho + p)H = -Q\dot{\varphi} \tag{2.8}$$

and the Klein-Gordon equation for $\varphi = \varphi(t)$

$$\ddot{\varphi} + 3H\dot{\varphi} = Q, \tag{2.9}$$

where

$$Q = \frac{[\rho - 3p]}{\sqrt{2(3+2\omega)}}. \tag{2.10}$$

When $Q \neq 0$, equations (2.8) and (2.9) indicate energy-transfer between the matter and scalar field. $Q = 0$ when $T \equiv T^a_a = 3p - \rho = 0$ (i.e., $3\overline{p} - \overline{\rho} = 0$). We shall assume that the matter satisfies the equation of state $\overline{p} = (\gamma - 1)\overline{\rho}$ (i.e., $p = (\gamma - 1)\rho$) where γ is a constant.

Finally, defining (Tabensky and Taub, 1973)

$$\rho_\varphi = p_\varphi = \frac{1}{2}\dot{\varphi}^2, \tag{2.11}$$

so that (2.9) becomes

$$\dot{\rho}_\varphi + 3(\rho_\varphi + p_\varphi)H = Q\dot{\varphi}, \tag{2.12}$$

we see that the massless scalar field is equivalent to a stiff perfect fluid ($\gamma_\varphi = 2$). Hence the model is equivalent to an interacting two-fluid model, one fluid of which is stiff (Mimosa and Wands, 1995b).

The study of interacting two-fluid models is very complicated. However, there are three special cases of interest in which there is no interaction between the two fluids. First, from equation (1.10), $Q = 0$ if $\rho = 3p$ ($\overline{\rho} = 3\overline{p}$); this can occur either for vacuum ($\overline{\rho} = \overline{p} = 0$; i.e., no matter present) or in the case of radiation ($\gamma = 4/3$). Second, in the case of stiff matter ($\gamma = 2$), the total energy density and pressure are given by

$$p_{tot} = p + p_\varphi = \rho + \rho_\varphi = \rho_{tot}, \tag{2.13}$$

and so the two-fluid model is equivalent to single stiff fluid model satisfying equations (1.13). Finally, the case in which the matter field is equivalent to a cosmological constant is tractable.

Although progress is only possible in these very special cases, these cases are nonetheless of particular physical importance. For example, cosmological models with matter are known to be asymptotic to vacuum models in a variety of circumstances (Wainwright and Ellis (WE), 1997), and radiation matter fields and a cosmological constant or a vacuum energy density are known to play an important rôle in the early Universe. In addition, stiff homogeneous perfect fluids represent (additional) homogeneous massless scalar fields and can formally model geometry effects [e.g., the shear scalar in a Bianchi I model in the conformally transformed Einstein frame behaves exactly like a stiff fluid (Mimosa and Wands, 1995b)]. Indeed, the long wave-length modes of a massless scalar field act like a stiff fluid, and a stiff fluid may also describe the evolution of an effectively massless field [including, in the context of superstring cosmology, the antisymmetric tensor field which appears in the low energy effective action (Copeland et al., 1994)], and if present they would be expected to dominate at early times in the Universe (over, for example, any short wave-length modes or any other matter fields with $\rho > p$).

In the next section we shall study the vacuum case by reviewing the asymptotic properties of stiff perfect fluid spatially homogeneous models (WE). Once this is done, we can determine the scalar field φ by integrating equation (2.12) whence we can determine the asymptotic properties of the scalar-tensor models from equations (2.2) and (2.3) (see section 4). The properties of models in which the matter field is a stiff perfect fluid can be deduced from the results in the vacuum case. The case in which the second fluid is radiation will be dealt with in subsection 3.2 and the case of a cosmological constant will be dealt with in subsection 3.3.

3 Results in General Relativity

The qualitative properties of orthogonal spatially homogeneous (OSH) perfect fluid models with an equation of state $p = (\gamma - 1)\rho$ within GR have been studied by Wainwright and collaborators (see WE and references within). Indeed, the governing equations of these models reduce to a (finite) n-dimensional polynomial system of autonomous ordinary differential equations. Utilizing an orthonormal frame approach and introducing an expansion-normalized (and hence dimensionless) set of variables, it was shown that one differential equation (for the expansion or the Hubble parameter) decouples from the remaining equations, allowing for the study of a "reduced" (i.e., $(n-1)$-dimensional) system of ordinary differential equations. In particular, it was proven that all of the singular points of the "reduced" dynamical system correspond to exact (time-evolving) solutions admitting a homothetic vector (Hsu and Wainwright, 1986). Therefore, these (transitively) self-similar cosmological models play an important rôle in describing the asymptotic behaviour of the spatially homogeneous cosmologies. The dynamics of the more general Bianchi models is complicated by the fact that there exist lower-dimensional attractors (that are not simple singular points) in Bianchi types VIII and IX models (which determine their early time behaviour) and the phase space of models of types VII_0, VIII and IX are not compact (which affects the determination of their late time behaviour).

The value $\gamma = 2$, of interest in the study of stiff perfect fluids, is a bifurcation value for γ in the reduced dynamical system; consequently models with $\gamma = 2$ may have different qualitative properties to models with $\gamma < 2$. A complete discussion of the case $\gamma = 2$ has yet to be given, so we begin with a review of this case.

3.1 Stiff perfect fluids in GR

The finite singular points (and their stability) of the reduced dynamical system in the case $\gamma = 2$ has been investigated by Wainwright and collaborators (WE). Indeed, *all* non-tilting spatially homogeneous solutions of the Einstein field equations with a perfect fluid with $\gamma = 2$ (and $\rho > 0$) as source which admit a four-dimensional similarity group acting simply transitively on spacetime are listed in table 9.2 in WE. In particular, the flat isotropic $\gamma = 2$ solution (FL) is given by

$$\begin{aligned} ds^2_{FL} &= -dt^2 + t^{2/3}(dx^2 + dy^2 + dz^2), \\ \rho &= \frac{1}{3}t^{-2}, \end{aligned} \tag{3.1}$$

which is an attractor in the class of isotropic models and has important physical applications, and the Jacobs stiff perfect fluid solutions $\mathcal{J}$ given by

$$
\begin{aligned}
ds^2_{\mathcal{J}} &= -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \\
\rho &= \frac{1}{2}(1-p^2)t^{-2},
\end{aligned}
\tag{3.2}
$$

where the two essential parameters are determined by

$$p_1 + p_2 + p_3 = 1, \quad p^2 \equiv p_1^2 + p_2^2 + p_3^2 < 1.$$

All $\gamma = 2$ Bianchi I solutions are Jacobs self-similar solutions and each solution corresponds to a singular point on the 'Jacobs Disc'. These solutions play an important role in describing the qualitative properties of classes of Bianchi models. Theorem 9.2 in WE states that all known $\gamma = 2$ solutions correspond to singular points on the Jacobs Disc or the (vacuum) Kasner Ring (see WE p.199 for precise definitions of these sets).

In particular, in Wainwright and Hsu (1989) OSH models of type A were studied. Although their analysis was not conclusive, they showed (Proposition 4.1) that there exists a strictly monotonic function on each of the $\gamma = 2$ Bianchi invariant sets. The singular points occur on the Jacobs Disc in the Bianchi I invariant set (corresponding to Jacobs stiff fluid solution) or the Kasner Ring. The stability of the singular points on the Jacobs Disc, complicated by the existence of two zero eigenvalues (of the five-dimensional set), was discussed on p.1426 in Wainwright and Hsu (1989), and the stability of those on the Kasner Ring was discussed on p.1427 where it was shown that a subset of these points act as sources in the various Bianchi type VIII and IX invariant sets. In addition, OSH models of type B (not including the exceptional Bianchi type $\mathrm{VI}_{-1/9}$ case) were considered in Hewitt and Wainwright (1993), where it was shown (Proposition 5.3) that all such Bianchi models with $\gamma = 2$ are asymptotic in the past to a Jacob's Bianchi I model and asymptotic to the future to a vacuum plane wave state. All self-similar vacuum solutions, including the plane wave solutions and various forms of flat spacetime, are listed in table 9.1 in WE. For example, the one-parameter Bianchi type VII_h plane wave solution is given by

$$ds^2_{PW} = -dt^2 + t^2dx^2 + t^{2r}e^{2rx}\{e^{\beta}[\cos v\ dy + \sin v\ dz]^2 + e^{-\beta}[\cos v\ dz - \sin v\ dz]^2\}, \tag{3.3}$$

where $v \equiv b(x + lnt)$ and the constants in (3.3) satisfy

$$b^2 \sinh^2 \beta = r(1-r);\ b^2 = r^2/h, \quad 0 < r < 1,$$

where $h > 0$ is the group parameter. The case $r = 1$ $(\beta = 0)$ gives the Bianchi VII_h version of the Milne model; the Milne form of flat spacetime is given by

$$ds^2_M = -dt^2 + t^2[dx^2 + e^{2x}(dy^2 + dz^2)]. \tag{3.4}$$

As mentioned earlier, all of the singular points correspond to transitively self-similar cosmological solutions. In particular, the homothetic vectors corresponding to the metrics (3.1)–(3.4) are given by

$$\begin{aligned}
X_{FL} &= t\frac{\partial}{\partial t} + \frac{2}{3}\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\}, \\
X_{\mathcal{J}} &= t\frac{\partial}{\partial t} + (1-p_1)x\frac{\partial}{\partial x} + (1-p_2)y\frac{\partial}{\partial y} + (1-p_3)z\frac{\partial}{\partial z}, \\
X_{PW} &= t\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \\
X_M &= t\frac{\partial}{\partial t}.
\end{aligned} \tag{3.5}$$

3.2 Non-interacting perfect fluid and stiff matter

The analysis of the qualitative properties of two *non-interacting* perfect fluid OSH models was presented in Coley and Wainwright (1992). The situation of interest here corresponds to the case in which the first fluid is stiff ($\gamma_1 = 2$) and the second is radiation ($\gamma_2 = 4/3$). Defining the radiation density and the stiff matter density by ρ_r and ρ_s, respectively, where

$$\begin{aligned}
\rho_r &= \rho_{\gamma=4/3} \\
\rho_s &= \rho_{\gamma=2} = \rho_\varphi,
\end{aligned} \tag{3.6}$$

and introducing the new variable χ, defined by

$$\chi = \frac{\rho_r - \rho_s}{\rho_r + \rho_s}; \quad -1 \le \chi \le 1, \tag{3.7}$$

from the (separate) conservation laws (2.8) and (2.12) ($Q = 0$ from equation (2.10)), we obtain the following time evolution equation for χ:

$$\chi' \equiv \frac{1}{H}\dot{\chi} = 1 - \chi^2 \ge 0. \tag{3.8}$$

Hence, for ever-expanding models ($H > 0$), χ is monotonically increasing with

$$\lim_{t\to 0}\chi = -1, \quad \lim_{t\to\infty}\chi = +1. \tag{3.9}$$

This means that the corresponding cosmological models evolve from an initial state in which the stiff fluid ($\chi = -1$) dominates to a final state in which the radiation fluid is dominant ($\chi = +1$); i.e., the asymptotic behaviour of the two-fluid OSH models is described by the asymptotic behaviour of the associated single-fluid models. Therefore, the early time

behaviour of the GR model with a non-interacting stiff fluid and radiation or the equivalent massless scalar field model coupled to radiation, or the associated BDT model with radiation, can be deduced from the results of the previous subsection. The late-time behaviour of these models is dominated by the radiation.

3.3 Cosmological constant

For initially expanding GR spatially homogeneous models with matter and a positive cosmological constant, including the case of a minimally coupled massless scalar field, the late time behaviour is determined by the cosmic no-hair theorem (Wald, 1983); namely, all Bianchi models (except a subclass of type IX) are future-asymptotic to de Sitter spacetime (see also Coley and Wainwright, 1982).

The late time behaviour of ever-expanding spatially homogeneous models in BDT with matter and a positive constant vacuum energy density[2] [where the term $\phi\overline{R}$ in the action (2.1) becomes $\phi(\overline{R}+\lambda)$] can be determined directly from the cosmic no-hair theorem results of Guzman (1997); namely, all such models are future asymptotic to a flat, isotropic, power-law state (extended inflation; La and Steinhardt, 1989).[3] Isotropization and inflation in anisotropic scalar-tensor theories was discussed earlier by Pimentel and Stein-Schabes (1989).

4 Applications

To study the qualitative properties of OSH perfect fluid models (with an equation of state $p=(\gamma-1)\rho$) within scalar-tensor theories (with no potential), and particularly within BDT, expansional-normalized variables can be introduced and the resulting system of ordinary differential equations can be investigated (Billyard et al., 1998). In BDT it can be shown that again one differential equation decouples and the "reduced" finite-dimensional system of ordinary differential equations can be analysed (WE); the singular points of the reduced dynamical system again correspond to exact self-similar solutions (Coley and van den Hoogen, 1994). However, here we shall determine some of the more important qualitative properties directly by utilizing the results in the earlier sections and noting that solutions corresponding to singular points of the governing dynamical system can act as future and past attractors.

[2]Note that, unlike in GR, a cosmological constant is not identical to the presence of a vacuum energy in BDT.

[3]A similar scenario, referred to as hyperextended inflation, occurs in scalar-tensor theories of gravity with $\omega(\phi)$ (Steinhardt and Accetta, 1990).

First, we choose coordinates in which the OSH metric can be written as

$$ds^2 = -dt^2 + \gamma_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta; \quad \alpha = 1, 2, 3. \tag{4.1}$$

At the finite singular points of the reduced GR dynamical system (in expansion-normalized variables) it can be shown that (Wainwright and Hsu, 1989) $\theta \propto t^{-1}$, where θ is the expansion of the timelike congruences orthogonal to the surfaces of homogeneity. Defining the Hubble parameter by $H = \theta/3$, it follows that

$$H = H_0 t^{-1}. \tag{4.2}$$

Also, at the singular points we have that

$$\frac{\rho_\varphi}{H^2} = d^2, \tag{4.3}$$

where d^2 is a positive constant which can be determined from the generalized Friedmann equation. From the energy conservation equation we then find that $H_0 = 1/3$, so that

$$H = \frac{1}{3} t^{-1}, \quad \rho_\varphi = \frac{d^2}{9} t^{-2}. \tag{4.4}$$

Exact cosmological solutions corresponding to the singular points must obey equations (4.4).

In addition, these exact OSH solutions are transitively self-similar (Wainwright and Hsu, 1989); i.e., if g_{ab} represents the spacetime metric corresponding to such a solution then there exists a HV X satisfying

$$\mathcal{L}_X g_{ab} = 2 g_{ab}, \tag{4.5}$$

where $\mathcal{L}$ denotes Lie differentiation along X. The HVs corresponding to the exact solutions (3.1)–(3.4) were given by (3.5); we note that X is of the form

$$X = t \frac{\partial}{\partial t} + X^\alpha(x^\gamma) \frac{\partial}{\partial x^\alpha} \tag{4.6}$$

in all of these four cases. Indeed, we can show that in the coordinates (4.1) any such HV will always be of this form as follows.

First, since the energy-momentum tensor is of the form of a perfect fluid with four-velocity u^a, it follows from the Einstein field equations and equation (4.5) that

$$\mathcal{L}_X u^a = -u^a,$$

whence it follows that (Coley and Tupper, 1989)

$$\mathcal{L}_X H = -H. \tag{4.7}$$

Now, writing $X = X^0 \frac{\partial}{\partial t} + X^\alpha \frac{\partial}{\partial x^\alpha}$ and using (4.4), equation (4.7) implies that $X^0 \frac{\partial}{\partial t}(t^{-1}) = -t^{-1}$; i.e., $X^0 = t$. ¿From (4.1), the (0α)-components of (4.5) then trivially yield $X^\alpha = X^\alpha(x^\gamma)$, and we obtain the result (4.6).

Next, the scalar field $\varphi(t)$ in the Einstein frame is related to ρ_φ by equation (2.11), so that from (4.4) we obtain

$$\frac{d\varphi}{dt} = \frac{\sqrt{2}d}{3} t^{-1}. \tag{4.8}$$

The scalar field $\phi(t) = \phi(\overline{t}(t))$ in the Jordan frame is determined by equation (2.3), whence

$$\frac{1}{\phi}\frac{d\phi}{dt} = 2W(\phi)t^{-1}, \tag{4.9}$$

where $W(\phi) \equiv \pm\frac{d}{3}(2\omega(\phi)+3)^{-\frac{1}{2}}$. In the BDT, where ϕ is the BD scalar, we have $\omega(\phi) = \omega_0$ and so

$$W(\phi) = \varpi \equiv \pm\frac{d}{3}(2\omega_0 + 3)^{-\frac{1}{2}}, \text{ a constant.} \tag{4.10}$$

Now, the scalar-tensor metric $\overline{g}_{ab}$ in the Jordan frame is related to the GR metric g_{ab} in the Einstein frame by equation (2.2), so that

$$\mathcal{L}_X \overline{g}_{ab} = X(\phi^{-1})g_{ab} + \phi^{-1}\mathcal{L}_X g_{ab},$$

whence from equations (4.5), (4.6) and (4.9) we obtain

$$\begin{aligned} \mathcal{L}_X \overline{g}_{ab} &= -t\frac{\dot{\phi}}{\phi}\phi^{-1}g_{ab} + 2\phi^{-1}g_{ab} \\ &= 2[1 - W(\phi)]\overline{g}_{ab}. \end{aligned} \tag{4.11}$$

Therefore, for a scalar-tensor theory with $\omega = \omega(\phi)$, X is a conformal Killing vector for the corresponding exact solution in the scalar-tensor theory.

In the particular case of BDT (only), $1 - W(\phi) = 1 - \varpi$, a constant, and hence X is in fact a HV. Consequently, the associated exact solution in BDT, which can act as a past or future attractor, is again transitively *self-similar*. The BD scalar field can be obtained from equation (4.9), and is given by (in the time coordinate t)

$$\phi = \phi_0 t^{2\varpi}, \tag{4.12}$$

where ϕ_0 is an integration constant. From equation (4.9) we note that the form of $\phi(t)$ in solutions corresponding to the singular points is well defined for all $t > 0$, and hence the conformal transformation (2.2) is regular and we can therefore deduce the qualitative

behaviour of the scalar-tensor models (as $t \to 0^+$ and $t \to \infty$) directly from their GR counterparts.

Finally, we note that for the degenerate case in GR in which H is a constant (e.g., de Sitter spacetime or Minkowski spacetime), equation (4.2) and the ensuing analysis does not follow. We note that this is related to the special case above in which the BD constant ω_0 is such that $\overline{\omega} = 1$, whence the vector field X in (4.8) becomes a Killing vector and the associated GR spacetime is (four-dimensionally) homogeneous (see Kramer et al., 1980.)

4.1 Massless scalar field in GR

The form of the geometry in the exact solutions corresponding to singular points of the governing dynamical system for stiff perfect fluids was discussed in section 3. Consequently, to determine the asymptotic properties of massless scalar field models in GR we simply need to determine the form for the scalar field φ in these models, which is obtained by integrating equation (4.8), viz.,

$$\varphi(t) = \varphi_0 + \frac{\sqrt{2}d}{3} \ln t \tag{4.13}$$

(see also Belinskii and Khalatnikov, 1973).

4.2 Brans-Dicke Theory

To determine the asymptotic properties of BDT spatially homogeneous models we shall exploit their formal equivalence to GR models and use the preceding results. In order to present some particular results we shall consider the exact GR solutions (3.1)–(3.4).

In the BDT (in the Jordan frame) $\phi(t(\overline{t}))$ is given by equation (4.12) and from equations (2.2) and (4.1) we find that the associated BD metric is given by (in the time coordinate t)

$$d\overline{s}^2 \equiv d\overline{s}^2_{BD} = -\phi_0^{-1} t^{-2\overline{\omega}} dt^2 + \phi_0^{-1} t^{-2\overline{\omega}} \gamma_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta. \tag{4.14}$$

Defining a new time coordinate T by ($\overline{\omega} \neq 1$)

$$T = ct^{1-\overline{\omega}}, \tag{4.15}$$

where $c \equiv \phi_0^{-\frac{1}{2}}(1-\overline{\omega})^{-1}$, we obtain

$$d\overline{s}^2 = -dT^2 + C^2 T^{-\frac{2\overline{\omega}}{1-\overline{\omega}}} \gamma_{\alpha\beta}(t(T), x^\gamma) dx^\alpha dx^\beta, \tag{4.16}$$

where $t(T) = (T/c)^{1/(1-\overline{\omega})}$ and $C^2 \equiv \phi_0^{-1} c^{2\overline{\omega}/(1-\overline{\omega})}$. Finally, the BD scalar field is given by

$$\phi(T) = C^{-2} T^{\frac{2\overline{\omega}}{1-\overline{\omega}}}. \tag{4.17}$$

¿From the form of $\phi(T)$ ($\overline{\omega} \neq 1$) we note that the transformations between BDT and GR are non-singular as $T \to 0^+$ and $T \to \infty$ and hence we can deduce the asymptotic properties of the BDT models directly from the corresponding models in GR.

1: For the flat isotropic FL metric (3.1), the BD metric (4.16) becomes (after a constant rescaling of the spatial coordinates)

$$d\overline{s}^2_{FL} = -dT^2 + T^{\frac{2(1-3\overline{\omega})}{3(1-\overline{\omega})}} (dX^2 + dY^2 + dZ^2). \tag{4.18}$$

We deduce that the exact, flat (non-inflationary) isotropic BD solution (4.18) and (4.17) is an attractor in the class of isotropic models in BDT (see Holden and Wands, 1998). This vacuum BD solution (where $d^2 = 3$ is determined from the generalized Friedmann equation) was first obtained by O'Hanlon and Tupper (1972). We note that for large values of ω_0, $2(1-3\overline{\omega})/3(1-\overline{\omega}) \approx \frac{2}{3}\left(1 \pm \frac{d\sqrt{2}}{3\sqrt{\omega_0}}\right) \approx \frac{2}{3}$; indeed, as $\omega_0 \to \infty$, $\overline{\omega} \to 0$ and we formally recover the GR solution in this limit.

2: For the Jacobs stiff perfect fluid solutions (3.2), the associated BD metric (4.16) becomes (after a constant rescaling of each spacelike coordinate)

$$d\overline{s}^2_{\mathcal{J}} = -dT^2 + T^{2q_1} dX^2 + T^{2q_2} dY^2 + T^{2q_3} dZ^2, \tag{4.19}$$

where

$$q_i = \frac{p_i - \overline{\omega}}{1 - \overline{\omega}}; \quad \alpha = 1, 2, 3$$

(Brans and Dicke, 1961; Belinskii and Khalatnikov, 1973). The Bianchi type I BD solutions (4.19) and (4.17) therefore act as attractors for a variety of OSH Bianchi models (see subsection 3.1). In particular, all non-exceptional, initially expanding Bianchi type B BD models are asymptotic in the past to this BD solution. The metric (4.19) reduces to metric (4.18) in the isotropic case in which all of the q_i are equal (i.e., $p_i = 1/3$, $i = 1, 2, 3$).

3: From subsection 3.1 we can conclude that all Bianchi models of type B in BDT are asymptotic to the future to a vacuum plane wave state. For example, from (3.3) we obtain the following BD Bianchi type VII$_h$ plane wave solution (after a constant rescaling of the 'y' and 'z' coordinates)

$$d\overline{s}^2_{PW} = -dT^2 + D^2 (T^2 dx^2 + T^{2\frac{(r-\overline{\omega})}{1-\overline{\omega}}} e^{2rx} \{ e^{\beta} [\cos v\, dY - \sin v\, dZ]^2 + e^{-\beta} [\cos v\, dZ - \sin v\, dY]^2 \}), \tag{4.20}$$

where $D \equiv C c^{-1/(1-\overline{\omega})}$ and $v = b(x + \frac{1}{1-\overline{\omega}} ln[T/c])$ (and all other constants are defined as before). Metric (4.20) can be simplified by a redefinition of the 'x' coordinate. The BD scalar field is given by (4.17). This exact BD plane wave solution is believed to be new.

¿From the discussion of the asymptotic properties of stiff perfect fluid models in GR given in subsection 3.1 and as summarized in the Introduction, we can now deduce the asymptotic properties of spatially homogeneous cosmological models in BDT (in particular, see cases 2 and 3 above). Indeed, all of the GR results reviewed in the Introduction have BDT analogues. For example, all orthogonal Bianchi type B BDT models, except for a set of measure-zero, are asymptotic to the future to a vacuum plane-wave state (see, for example, equation (4.19)). One immediate consequence of this result, since Bianchi models of type B constitute a set of positive measure in the set of spatially homogeneous initial data, is that a recent conjecture that the initial state of the pre-big-bang scenario within string theory generically corresponds to the Milne (flat spacetime) universe (Veneziano, 1991; Gasparini and Veneziano, 1993; Clancy et al., 1998) is unlikely to be valid (recall that past asymptotic behaviour in pre-big-bang cosmology corresponds to future asymptotic states in classical cosmological solutions).

4: Finally, in the special case of de Sitter spacetime

$$ds^2_{dS} = -dt^2 + e^{2H_0 t}(dx^2 + dy^2 + dz^2), \tag{4.21}$$

we have that $H(t) = H_0$, a constant, and we cannot use the analysis of subsection 4.1 (compare with equation (4.2); e.g., equations (4.8) and (4.9) are not valid). In this case we have that

$$\frac{1}{2}\dot{\varphi}^2 = \rho_\varphi = d^2 H_0^2, \tag{4.22}$$

and hence from equations (2.3) and (4.10) we obtain

$$\frac{\dot{\phi}}{\phi} = -6|\overline{\omega}|H_0,$$

whence we find that

$$\phi(t) = \overline{\phi}_0 \exp\{-6|\overline{\omega}|H_0 t\}, \tag{4.23}$$

where $\overline{\phi}_0$ is an integration constant. Using (2.2) to obtain the metric in the Jordan frame, and introducing the new time coordinate $T \propto \exp(3|\overline{\omega}|H_0 t)$, we obtain (after a constant rescaling of the spatial coordinates)

$$d\overline{s}^2_{dS} = -dT^2 + T^{2\left(1+\frac{1}{3|\overline{\omega}|}\right)}(dX^2 + dY^2 + dZ^2), \tag{4.24}$$

where $\phi(T) \propto T^{-2}$. This flat, isotropic, power-law BD solution is clearly inflationary ($1 + 1/3|\overline{\omega}| > 1$) (cf. extended inflation, La and Steinhardt, 1989).

4.3 Scalar-tensor theories

In the same way we can study the asymptotic properties of scalar-tensor theory models with action conformally related to (2.1), where for general $\omega = \omega(\phi), \phi(t)$ would be determined

from (4.9) (and not given by equation (4.12)). The asymptotic properties of more general scalar-tensor theories can be studied in a similar way (cf. Billyard et al., 1998).

5 Discussion

The results presented in this paper are the first concerning the generic asymptotic properties of spatially homogeneous models in scalar-tensor theories of gravity. However, some care is needed in interpreting these results and they must be applied in concert with special exact solutions and the analysis of specific but tractable classes of models (cf. Mimosa and Wands, 1995b) to build up a complete cosmological picture.

First, the GR analysis (in the Einstein frame) is incomplete in that the phase-space in some Bianchi classes is not compact and the Hubble parameter can become zero (and hence the expansion-normalized variables become ill-defined). Second, in scalar-tensor theories of gravity (in the Jordan frame) it is known that there exist solutions which do not have an initial singularity but have a 'bounce' (at which $H = 0$) – typically this occurs for negative values of ω; e.g., the initial singularity is avoided in BDT if $\omega_0 < -4/3$ (Nariai, 1972). Since there is always an initial singularity in the Einstein frame, such an 'avoidance of a singularity' is due to the properties of the transformations (2.2) and (2.3); for example, Mimosa and Wands (1995b) describe a set of models that reach an anisotropic singularity in a finite time in the Einstein frame which correspond to non-singular and shear-free evolution in infinite proper time in the Jordon frame.

Consequently, although the asymptotic results presented here are generally valid, the full dynamical properties of the scalar-tensor models, including their global features and their physical interpretation, are determined from these asymptotic results and the properties of the transformations and how solutions are matched together to construct the complete dynamical picture.

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References

T. Applequist, A. Chodos and P.G.O. Freund, 1987, *Modern Kaluza-Klein Theories*

(Redwood City: Addison-Wesley).

J.D. Barrow, 1993, Phys. Rev. D. **47**, 5329.

J.D. Barrow, 1996, MNRAS **282**, 1397.

J.D. Barrow and K.E. Kunze, 1998, gr-qc/9807040.

J.D. Barrow and J.P. Mimosa, 1994, Phys. Rev. D. **50**, 3746.

J.D. Barrow and P. Parsons, 1997, Phys. Rev. D. **55**, 1906.

V.A. Belinskii and I.M. Khalatnikov, 1973, Sov. Phys. JETP **36**, 591.

A.P. Billyard, A.A. Coley and J. Ibáñez, 1998, Phys. Rev. D. **59**, 023507.

C. Brans and R.H. Dicke, 1961, Phys. Rev. **124**, 925.

R. Carretero-Gonzalez, H.N. Nunez-Yepez and A.L. Salas Brito, 1994, Phys. Letts. A. **188**, 48.

P. Chauvet and J.L. Cervantes-Cota, 1995, Phys. Rev. D. **52**, 3416.

D. Clancy, J.E. Lidsey and R. Tavakol, 1998, gr-qc/9802052

A.A. Coley and R.J. van den Hoogen, 1994, in *Deterministic Chaos in General Relativity*, ed. D. Hobill et al. (Plenum).

A.A. Coley and B.O.J. Tupper, 1989, J. Math Phys. **30**, 2618.

A.A. Coley and J. Wainwright, 1992, Class. Quantum Grav. **9**, 651.

E.J. Copeland, A. Lahiri and D. Wands, 1994, Phys. Rev. D. **50**, 4868.

M. Gasperini and G. Veneziano, 1993, Astropart. Phys. **1**, 317.

M.B. Green, J.H. Schwarz and E. Witten, 1988, *Superstring Theory* (Cambridge: Cambridge University Press).

L.E. Gurevich, A.M. Finkelstein and V.A. Ruban, 1973, Ap. Sp. Sci. **22**, 231.

E. Guzman, 1997, Phys. Letts. B. **391**, 267.

C. Hewitt and J. Wainwright, 1993, Class. Quantum Grav. **10**, 99.

D.J. Holden and D. Wands, 1998, gr-qc/9803021.

S.J. Kolitch and D.M. Eardley, 1995, Ann. Phys. (N.Y.) **241**, 128.

D. Kramer, H. Stephani, E. Herit and M.A.H. MacCallum, 1980, *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press).

D. La and P.J. Steinhardt, 1989, Phys. Rev. Letts. **62**, 376.

J.E. Lidsey, 1998, preprint.

D. Lorenz-Petzold, 1984, in *Solutions to Einstein's Equations: Techniques and Results*, Proc. of the International Seminar, Retzbach (Germany), eds. C.H. Hoenselaers and W. Dietz, Lecture Notes in Physics volume 205 (Springer-Verlag, Berlin).

J.P. Mimoso and D. Wands, 1995a, Phys. Rev. D. **52**, 5612.

J.P. Mimoso and D. Wands, 1995b, Phys. Rev. D. **51**, 477.

H. Nariai, 1968, Prog. Theoret. Phys. **40**, 49.

H. Nariai, 1972, Prog. Theoret. Phys. **47**, 1824.

J. O'Hanlon and B.O.J. Tupper, 1972, Il Nuovo Cimento B **7**, 305.

L.O. Pimentel and J. Stein-Schabes, 1989, Phys. Letts. B. **216**, 27.

V.A. Ruban, 1977, Sov. Phys. JETP **45**, 629.

A.A. Serna and J.M. Alimi, 1996, Phys. Rev. D. **53**, 3074 and 3087.

P.J. Steinhardt and F.S. Accetta, 1990, Phys. Rev. Letts. **64**, 2740.

R. Tabensky and A. H. Taub, 1973, Commun. Math. Phys. **29**, 61.

G. Veneziano, 1991, Phys. Lett. **B265**, 287.

R. M. Wald, 1983, Phys. Rev. D. **28**, 2118.

J. Wainwright and G.F.R. Ellis, 1997, *Dynamical Systems in Cosmology* (Cambridge: Cambridge University Press).

J. Wainwright and L. Hsu, 1989, Class. Quantum Grav. **6**, 1409.

C.M. Will, 1993, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge).